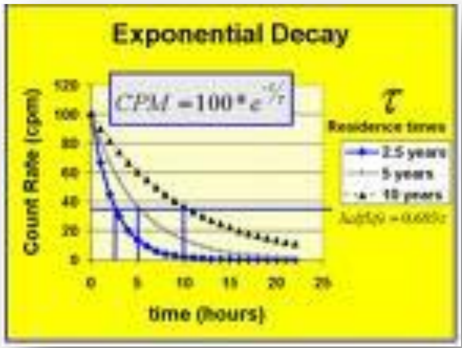

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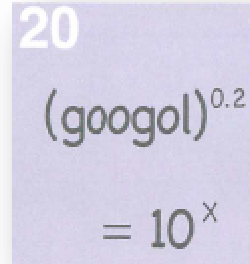

Good morning! Welcome to another Friday!

Let's Talk About Growth and Decay

Let's take a look at Section 6.2 in our textbook. Differential Equations: Growth and Decay. Knowing how to solve a differential equation opens up the possibility of solving problems where you know the rate of change, or derivative, of a function and you want to work backwards to find the original function.

Are There Any Techniques For Solving Differential Equations?

Before we get to a real life application of what we're learning let's stop and ask a very practical question: Are there any techniques for solving a differential equation? I mean, there are all these *x*'s and *y*'s and *y* primes and *y* double primes. Geez, whatever happened to adding 2 to both sides, dividing by 4, and solving for *x*?

You remember Algebra, where you spent all that time learning how to solve equations that ended up with *x* and *y* equal to a number. Well now we're solving differential equations with derivatives as unknowns and solving for functions of *x* instead of just *x*. We're ending up with an equation as a general solution.

Then if we want to, which we usually always do, we can select an initial condition and substitute it into the general solution to solve for that constant *C*, and that's when we get a particular solution which is just a plain old equation with an *x* and *y* as unknowns.

Don't you sometimes wish for the good old days? Well, don't give up yet. There are different ways of solving differential equations. They're definitely not as neat and clean as the ones you're used to. But the ones we'll be dealing with here will be solvable (well that's a relief), and we're about to learn a technique.

Separation of Variables – A Technique for Solving Differential Equations

The technique is called "separation of variables." It's a fancy name for rewriting the equation with all the like variables on one side of the equation, like this:

$$y' = \frac{\sqrt{x}}{3y}$$
$$3yy' = \sqrt{x}$$

I just multiplied both sides by 3*y* so I had all the *y*'s and *y* primes on one side and the *x*'s on the other. Pretty simple, huh? That's separation of variables in a nutshell. So now, how do we solve it? It looks pretty messy, doesn't it? And what are we solving for? Our ultimate goal is usually a particular function or curve that represents the same relationship between *x* and *y* but doesn't have any derivatives in it. But we've got to solve for the general solution to that differential equation that we started off with first.

Once I know the general solution, then if I know a particular *x* and *y*, I can single out which one of those many, many curves is the one that solves my problem. It will tell me the curve I'm looking for. That's usually the ultimate goal.

So we're trying to first find the general solution to the differential equation. Once we've got all the variables on their own sides, what do we do then? If we want to get rid of the derivatives and go backwards to the original function, what do we do? We integrate! So that's what we'll do:

$$y' = \frac{\sqrt{x}}{3y}$$
$$3yy' = \sqrt{x}$$
$$\int 3yy'\,dx = \int \sqrt{x}\,dx$$
$$\frac{3y^2}{2} = \frac{2}{3}x^{\frac{3}{2}} + C_1$$
$$9y^2 - 4x^{\frac{3}{2}} = C$$

Make sure you followed every step. It's a simple integration problem. Our end result is a family of curves, or functions, that solve the original differential equation we started with.

How Would We Use Differential Equations in Real Life?

So what kinds of real life applications use differential equations? Have you ever heard someone say "At this rate I'll never have enough money to pay for that car?" or "At this rate the population of the earth will reach 6 billion in the year xxxx?" or "At this rate I'll never finish reading these Comments?"

What do these real life situations have in common? These applications might be handy in predicting things. They all involve *rates*, which is a real obvious application to anything having to do with derivatives. But what else is common to them? It's more subtle. The unknown of the rate is the same unknown of the result. Huh? Let me say it another way. The variable of the rate is the same variable of the result.

The Rate of Change of the Unknown is Proportional to Itself

For example, the rate of saving *money* is going to result in not enough *money*, the rate of the *population* growth is going to result in *population*, the rate of reading *words* is going to result in a total number of *words*. There's no x's, there are just y's.

What we have here is a rate of change of an unknown that is proportional to itself. For instance, the growth of a population is a function of how many people you start off with, agree? A city with 100 people is going to grow a lot slower than a city with a million people. Isn't the time it takes you to finish these Comments, not only dependent on the rate of how many words you read per minute, but also on the number of words in the Comments?

The rate of the unknown is proportional to itself. The rate it takes you to finish is proportional to the number of words in the Comments.

$$\frac{dy}{dt} = ky$$

Notice I replaced x with t, because we're dealing with rates that have to do with time. The k stands for a constant. It's called a *constant of proportionality,* which makes sense.

How Can We Model Real Life with a Formula?

So here we are with a real life phenomena. The physicists and economists and biologists and whoever else needs it, came to the mathematicians with this phenomena. And they said, "Here this is the situation: we have this relationship that we have observed and we want to make predictions and do other things with this relationship. Can you write a formula that will model this relationship of a rate that is proportional to itself?"

They wanted a formula because the equation I wrote above is a differential equation; it has a derivative in it. They wanted a way to get from the rate or derivative to the variables, t and y, by themselves, without the derivative in the equation. So the mathematicians looked at the equation above and started off with a simpler one:

$$\frac{dy}{dt} = y$$

Now there's a conundrum. What derivative equals itself? Hmmm. I can see light bulbs lighting up. In a previous discussion we made a remarkable discovery about the exponential function with the special base $e = 2.71828$. We found out that the function

$$y = f(x) = e^x$$

has a special relationship to its own derivative, namely

$$\frac{dy}{dx} = e^x = y$$

Since we want to talk about applications where the independent variable is time, let's change *x* to *t*:

$$\frac{dy}{dt} = e^t = y$$

or, more simply,

$$\frac{dy}{dt} = y\,.$$

So what have the mathematicians discovered here? They have actually found the general solution to the equation

$$y' = y$$

Let's Test $y = e^t$ to See If It Works

Plug it in to test it:

$$\frac{d}{dt}e^t = e^t$$
$$e^t = e^t$$

Of course! The derivative of e^t equals itself. And it's the only function where that's true. So $y(t) = e^t$ is the general solution to the simpler version of the differential equation that the outsiders brought to the mathematicians.

Let's put a constant in front of *y* and see if $y(t) = e^t$ works for a constant. Why do we want to do that? Because that's the format of the phenomena the outsiders want us to model. Let's use $k = 2$ as the constant of proportionality:

$$\frac{dy}{dt} = 2y$$

Let's plug e^t in and try it:

$$y' = 2y$$
$$\frac{d}{dt}e^t = 2e^t$$
$$e^t = 2e^t$$

Rats! It doesn't work. So the mathematicians went back to the drawing board. They did some experimenting with different functions.

They Finally Found One That Did Work

Finally they came up with this function:

$$y(t) = Ce^{kt}$$

where k and C are constants. To find whether this function satisfies the above differential equation, we will need to compute its derivative, which we do using the Chain Rule. (We need to use the Chain Rule because the function above has an expression $u = kt$ in the exponent, not just t alone.)

$$\frac{dy}{dt} = \frac{d}{dt}\left(Ce^{kt}\right) = C\frac{de^{kt}}{dt} = C\frac{de^u}{du}\frac{du}{dt} = Ce^u k = k\left(Ce^{kt}\right) = ky$$

Notice that we just showed that **the function $y(t) = Ce^{kt}$ will satisfy the equation**

$$\frac{dy}{dt} = ky$$

Now that's a big deal! We can go to those outsiders and say Eureka! We've got your model or formula! It's $y(t) = Ce^{kt}$**! You can now do your predicting or whatever else you want to do with it. You can use your values for C and k and plug in your particular y and t values and solve for your particular solution equations.**

So, if we pick $k = 2$, the model should satisfy that differential equation we picked in the first place. Let's try it just for the heck of it. Let's plug in the model and see if the derivative of the model is equal to 2 times the model.

$$y' = 2y$$
$$\frac{d}{dt}(Ce^{2t}) = C\frac{de^{2t}}{dt} = C\frac{de^u}{du}\frac{du}{dt} = Ce^u 2 = 2(Ce^{2t})$$

It worked this time, when it didn't work when we just used $y(t) = e^t$! Bravo!

Let's Use It to Predict Something

So how would a scientist use this Exponential Growth and Decay Model $y(t) = Ce^{kt}$ to predict something? Let's say they know that plutonium decays at a rate proportionate to how much was released. They know that its half-life – the number of years required for half of the atoms in a sample to decay – is 24,100 years and they know how many grams were released. That formula will tell them the constant of proportionality and predict how many years it's going to take for the number of grams to decay down to let's say 1 gram just by them plugging in how many grams were released. I think that's pretty swift. Let's see how it does that:

The model equation is:

$$y(t) = Ce^{kt}$$

We're solving for t when $y = 1$ so we need to find C and k first from what we know. Let's say 10 grams of plutonium were released. That means $y = 10$. To find the constant of proportionality C we set $t = 0$ for the initial condition. And we solve for C:

$$\begin{aligned} 10 &= Ce^{k(0)} \\ &= Ce^{0} \\ 10 &= C \end{aligned}$$

So our constant of proportionality is 10. Now we've got to solve for k. Let's say we set $t = 24{,}100$ (that's its half-life). How many grams would be left? Wouldn't $y = 5$ by then? Yes, it would. There would be 5 grams of plutonium left after 24,100 years. So let's use that equation to solve for k:

$$\begin{aligned} &5 = 10e^{k(24{,}100)} \\ &\frac{1}{2} = e^{24{,}100k} \\ &\frac{1}{24{,}100}\ln\frac{1}{2} = k \\ &-0.000028761 \approx k \end{aligned}$$

So now we can solve for t when 1 gram is left in our original equation because we know y and C and k. Let's do it:

$$1 = 10e^{-0.000028761t}$$

$$\frac{1}{10} = e^{-0.000028761t}$$

$$\ln\frac{1}{10} = -0.000028761t$$

$$\frac{\ln\frac{1}{10}}{-0.000028761} = t$$

$$\frac{-2.302585}{-0.000028761} = t$$

$$80{,}059 \approx t$$

So it's going to take 80,059 years for the 10 grams to decay to 1 gram and we found that out by using the Exponential Growth and Decay model formula that the mathematicians so cleverly came up with in their calculations. Notice, the mathematicians had eliminated all of the derivatives from the calculations. You didn't have to solve for any derivatives to find the answer to the problem we just did.

The Exponential Growth and Decay Model Summary

The book just gives you the formula. So now you should have an idea of how we got to the formula. It all has to do with that wild and crazy number *e* and the fact that it has that special relationship with the derivative of e^x equaling itself. Now you should know what it means when people say "My goodness, that thing is growing EXPONENTIALLY!"

Here's a summary statement of what the mathematicians discovered:

> $y(t) = Ce^{kt}$ **is a solution to the differential equation** $\frac{dy}{dt} = ky$.

This is true for any value of the constant *C*.

Understanding this material is a matter of carefully going through each example in the book with paper and pencil so you can see how they arrive at each step in the process.

Videos

I have two videos for you today. Both of them show examples of solving first order separable differential equations which are the type of differential equations we are dealing with in Chapter 6. They are differential equations that can be solved by separation of variables. I'd like you to take a look at them before you try the homework problems.

The video with Patrick has an error in it at the end. You should recognize it. After he exponentiates both sides he forgets to cancel out the natural logarithm ln from the right side of the equation. There should be no ln in the final solution. There's a popup message pointing out the error.

Wrap Up

That's all I have for now. It's time for you to go to the Current Assignments page and get started.

Have a great day!