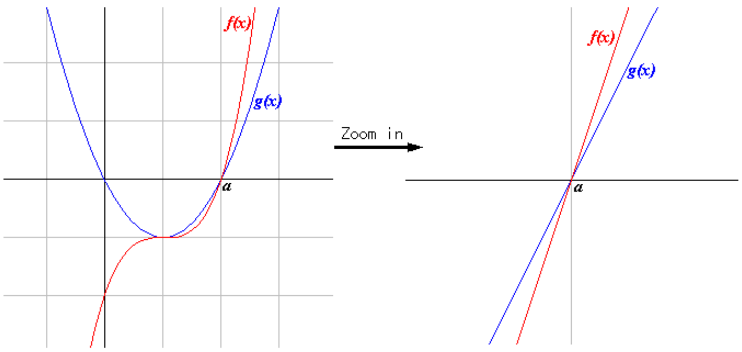


The graph on the right is an enlargement of the graph on the left at *x* = *a*. *f* (*x*) and *g*(*x*) are observed as straight lines. Then slopes of *f* (*x*) and *g*(*x*) are *f* '(*a*) and *g*'(*a*) respectively.

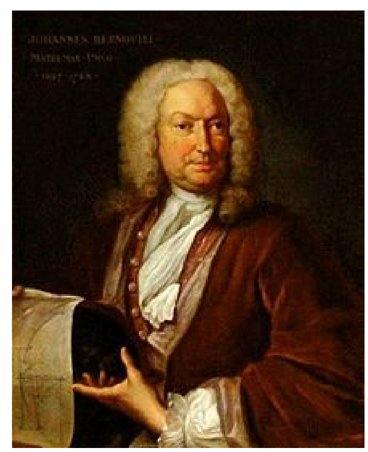
Good Morning!

Unit 7.7: Indeterminate Forms and L′Hôpital's Rule

Guillaume de l'Hôpital, after whom this rule is named

Today we move on to Section 7.7 on Indeterminate Forms and L′Hôpital's Rule. L'Hôpital is pronounced Loh-pee-tahl. So, who was this L′Hôpital character? Well, he was a rich French mathematician. In 1694 he forged a deal with Johann Bernoulli. The deal was L′Hôpital paid Bernoulli 300 Francs a year to tell him of his discoveries, which L′Hôpital described in his book on differential calculus.

In 1704, after L′Hôpital's death, Bernoulli revealed the deal to the world, claiming that many of the results in L′Hôpital's book were due to him. In 1922 texts were found that give support for Bernoulli. The widespread story that L′Hôpital tried to get credit for inventing L′Hôpital's Rule is false: he published his book anonymously, acknowledged Bernoulli's help in the introduction, and never claimed to be responsible for the rule. So, he was an okay guy. So, let's find out what this rule is all about.



Johann Bernoulli

Indeterminate Forms – What Do They Equal?

Back in Chapter 1 on Limits we saw methods for dealing with the following limits.

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4} \qquad \lim_{x \to \infty} \frac{4x^2 - 5x}{1 - 3x^2}$$

In the first limit if we plugged in $x = 4$ we would get 0/0 and in the second limit if we "plugged" in infinity we would get $\frac{\infty}{-\infty}$ (recall that as x goes to infinity a polynomial will behave in the same fashion that it's largest power behaves). Both are called **indeterminate forms**. In both cases, there are competing interests or rules and it's not clear which will win out.

In the case of 0/0 we typically think of a fraction that has a numerator of zero as being zero. However, we also tend to think of fractions in which the denominator is going to zero, as infinity or might not exist at all. Likewise, we tend to think of a fraction in which the numerator and denominator are the same, as one. So, which will win out? Or will neither win out and they all "cancel out" and the limit will reach some other value?

In the case of $\infty/-\infty$ we have a similar set of problems. If the numerator of a fraction is going to infinity, we tend to think of the whole fraction going to infinity. Also, if the denominator is going to infinity we tend to think of the fraction as going to zero. We also have the case of a fraction in which the numerator and denominator are the same (ignoring the minus sign) and so we might get -1. Again, it's not clear which of these will win out, if any of them will win out.

With the second limit, there is the further problem that infinity isn't really a number and so we really shouldn't even treat it like a number. Much of the time it simply won't behave as we would expect it to if it was a number.

This is the problem with indeterminate forms. It's just not clear what is happening in the limit. There are other types of indeterminate forms as well. Some other types are,

$$(0)(\pm\infty) \qquad 1^{\infty} \qquad 0^{0} \qquad \infty^{0} \qquad \infty-\infty$$

These all have competing interests or rules that tell us what should happen and it's just not clear which, if any, of the interests or rules will win out. The topic of this section is how to deal with these kinds of limits.

Sometimes We Can Factor an Indeterminate Form

As already pointed out we do know how to deal with some kinds of indeterminate forms already. For the two limits above we work them as follows.

$$\lim_{x\to 4}\frac{x^2-16}{x-4}=\lim_{x\to 4}\left(x+4\right)=8$$

$$\lim_{x\to\infty}\frac{4x^2-5x}{1-3x^2}=\lim_{x\to\infty}\frac{4-\frac{5}{x}}{\frac{1}{x^2}-3}=-\frac{4}{3}$$

In the first case, we simply factored, canceled and took the limit and in the second case we factored out an x^2 from both the numerator and the denominator and took the limit. Notice as well that none of the competing interests or rules in these cases won out! That is often the case.

But Sometimes Factoring Doesn't Work on Indeterminate Forms

So, we can deal with some of these. However, what about the following two limits.

$$\lim_{x \to 0} \frac{\sin x}{x} \qquad\qquad \lim_{x \to \infty} \frac{\mathbf{e}^x}{x^2}$$

The first one is a 0/0 indeterminate form, but we can't factor this one. The second is an ∞/∞ indeterminate form, but we can't just factor an x^2 out of the numerator. So, nothing that we've got in our bag of tricks will work with these two limits.

This is where the subject of this section comes into play.

L′Hôpital's Rule

Suppose that we have one of the following cases,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \qquad \text{OR} \qquad \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

where a can be any real number, infinity or negative infinity. In these cases, we have,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

So, L′Hôpital's Rule tells us that if we have an indeterminate form 0/0 or ∞/∞ all we need to do is differentiate the numerator and differentiate the denominator and then take the limit.

Let's Try Some Examples

***Example* 1.** Evaluate each of the following limits.

(a) $\lim_{x \to 0} \frac{\sin x}{x}$

We have already established that this is a 0/0 indeterminate form so let's just apply L′Hôpital's Rule.

Do this (showing each limit equals 0 justifies using L′Hopital's Rule which is required by AP Exam):

$$\lim_{x\to 0}\frac{\sin x}{x}$$
$$\lim_{x\to 0}\sin x = 0$$
$$\lim_{x\to 0} x = 0$$
Using L′Hopital's Rule
$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = \frac{1}{1} = 1$$

Not just this:

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = \frac{1}{1} = 1$$

(b) $\lim_{t\to 1}\frac{5t^4 - 4t^2 - 1}{10 - t - 9t^3}$

In this case, we also have a 0/0 indeterminate form and if we were really good at factoring we could factor the numerator and denominator, simplify and take the limit. However, that's going to be more work than just using L'Hôpital's Rule.

Do this (required by AP Exam):

$$\lim_{t\to 1}\frac{5t^4 - 4t^2 - 1}{10 - t - 9t^3}$$

$$\lim_{t\to 1} 5t^4 - 4t^2 - 1 = 0$$
$$\lim_{t\to 1} 10 - t - 9t^3 = 0$$

Using L′Hopital's Rule
$$\lim_{t\to 1}\frac{5t^4 - 4t^2 - 1}{10 - t - 9t^3} = \lim_{t\to 1}\frac{20t^3 - 8t}{-1 - 27t^2} = -\frac{12}{28} = -\frac{3}{7}$$

Not just this:

$$\lim_{t\to 1}\frac{5t^4-4t^2-1}{10-t-9t^3}=\lim_{t\to 1}\frac{20t^3-8t}{-1-27t^2}=\frac{20-8}{-1-27}=-\frac{3}{7}$$

(c) $\lim_{x\to\infty}\frac{\mathbf{e}^x}{x^2}$

This was the other limit that we started off looking at and we know that it's the indeterminate form ∞/∞ so let's apply L′Hôpital's Rule.

Do this:

$$\lim_{x\to\infty}\frac{e^x}{x^2}$$

$$\lim_{x\to\infty} e^x=\infty$$

$$\lim_{x\to\infty} x^2=\infty$$

Using L′Hopital's Rule

$$\lim_{x\to\infty}\frac{e^x}{x^2}=\lim_{x\to\infty}\frac{e^x}{2x}$$

$$\lim_{x\to\infty} e^x=\infty$$

$$\lim_{x\to\infty} 2x=\infty$$

Not just this:

$$\lim_{x\to\infty}\frac{\mathbf{e}^x}{x^2}=\lim_{x\to\infty}\frac{\mathbf{e}^x}{2x}$$

Now we have a small problem. This new limit is also an ∞/∞ indeterminate form. However, it's not really a problem. We know how to deal with these kinds of limits. Just apply L′Hôpital's Rule.

Do this:

$$\lim_{x\to\infty}\frac{e^x}{x^2}=\lim_{x\to\infty}\frac{e^x}{2x}$$

$$\lim_{x\to\infty}e^x=\infty$$

$$\lim_{x\to\infty}2x=\infty$$

Using L′Hopital's Rule

$$\lim_{x\to\infty}\frac{e^x}{x^2}=\lim_{x\to\infty}\frac{e^x}{2x}=\lim_{x\to\infty}\frac{e^x}{2}=\infty$$

Not just this:

$$\lim_{x\to\infty}\frac{e^x}{x^2}=\lim_{x\to\infty}\frac{e^x}{2x}=\lim_{x\to\infty}\frac{e^x}{2}=\infty$$

As you can see, sometimes we will need to apply L′Hôpital's Rule more than once.

With Some Manipulation We Can Handle Other Indeterminate Forms Too

L′Hôpital's Rule works great on the two indeterminate forms 0/0 and $\pm\infty/\pm\infty$. However, there are many more indeterminate forms out there as we saw earlier. Let's look at some of those and see how we deal with those kinds of indeterminate forms.

We'll start with the indeterminate form $(0)(\pm\infty)$.

***Example* 2.** Evaluate the following limit.

$$\lim_{x\to0^+}x\ln x$$

Note that we really do need to do the right-hand limit here. We know that the natural logarithm is only defined for positive *x* and so this is the only limit that makes any sense.

Now, in the limit, we get the indeterminate form $(0)(-\infty)$. L′Hôpital's Rule won't work on products; it only works on quotients. However, we can turn this into a fraction if we rewrite things a little.

$$\lim_{x\to0^+}x\ln x=\lim_{x\to0^+}\frac{\ln x}{1/x}$$

The function is the same, just rewritten, and the limit is now in the form $-\infty/\infty$ and we can now use L'Hôpital's Rule.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x}$$

$$\lim_{x\to 0^+} \ln x = -\infty$$

$$\lim_{x\to 0^+} 1/x = \infty$$

Using L′Hopital's Rule

$$\lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2}$$

Now, this is a mess, but it cleans up nicely.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$

In the previous example, we used the fact that we can always write a product of functions as a quotient by doing one of the following.

$$f(x)g(x) = \frac{g(x)}{1/f(x)} \qquad \text{OR} \qquad f(x)g(x) = \frac{f(x)}{1/g(x)}$$

Using these two facts will allow us to turn any limit in the form $(0)(\pm\infty)$ into a limit in the form 0/0 or $\pm\infty/\pm\infty$. Whether we end up with 0/0 or the other form $\pm\infty/\pm\infty$ after rewriting the limit depends on which function stays in the numerator and which gets moved down to the denominator.

Let's look at another example.

***Example* 3.** Evaluate the following limit.

$$\lim_{x\to -\infty} x\mathbf{e}^x$$

So, it's in the form $(\infty)(0)$. This means that we'll need to write it as a quotient. Moving the x to the denominator worked in the previous example so let's try that with this problem as well.

$$\lim_{x \to -\infty} x\mathbf{e}^x = \lim_{x \to -\infty} \frac{\mathbf{e}^x}{1/x}$$

Writing the product in this way gives us a product that has the form 0/0 in the limit. So, let's use L′Hôpital's Rule on the quotient. I'm going to skip writing all the steps required for the AP Exam to show you what happens with each use of L′Hopital's Rule.

$$\lim_{x \to -\infty} x\mathbf{e}^x = \lim_{x \to -\infty} \frac{\mathbf{e}^x}{1/x} = \lim_{x \to -\infty} \frac{\mathbf{e}^x}{-1/x^2} = \lim_{x \to -\infty} \frac{\mathbf{e}^x}{2/x^3} = \lim_{x \to -\infty} \frac{\mathbf{e}^x}{-6/x^4} = \cdots$$

Hmm…. This doesn't seem to be getting us anywhere. With each application of L′Hôpital's Rule we end up with another 0/0 indeterminate form and in fact the derivatives seem to be getting worse and worse. Also, note that if we simplified the quotient back into a product we would just end up with either $(\infty)(0)$ or $(-\infty)(0)$ and so that won't do us any good.

This does not mean however that the limit can't be done. It just means that we moved the wrong function to the denominator. Let's move the exponential function instead.

$$\lim_{x \to -\infty} x\mathbf{e}^x = \lim_{x \to -\infty} \frac{x}{1/\mathbf{e}^x} = \lim_{x \to -\infty} \frac{x}{\mathbf{e}^{-x}}$$

Note that we used the fact that:

$$\frac{1}{\mathbf{e}^x} = \mathbf{e}^{-x}$$

to simplify the quotient up a little. This will help us when it comes time to take some derivatives. The quotient is now an indeterminate form of $-\infty/\infty$ and using L′Hôpital's Rule gives:

$$\lim_{x \to -\infty} xe^{x} = \lim_{x \to -\infty} \frac{x}{e^{-x}}$$

$$\lim_{x \to -\infty} x = -\infty$$
$$\lim_{x \to -\infty} e^{-x} = \infty$$

Using L′Hopital's Rule

$$\lim_{x \to -\infty} \frac{x}{e^{-x}} = \lim_{x \to -\infty} \frac{1}{-e^{-x}} = 0$$

So, when faced with a product $(0)(\pm\infty)$ we can turn it into a quotient that will allow us to use L′Hôpital’s Rule. However, as we saw in the last example we need to be careful with how we do that on occasion. Sometimes we can use either quotient and in other cases only one will work.

Let’s now look at the indeterminate forms,

$$1^{\infty} \qquad 0^{0} \qquad \infty^{0}$$

These can all be dealt with in the following way, so we’ll just work one example.

***Example* 4.** Evaluate the following limit.

$$\lim_{x \to \infty} x^{\frac{1}{x}}$$

In the limit this is the indeterminate form x^{0}. We’re actually going to spend most of this problem on a different limit. Let’s first define the following.

$$y = x^{\frac{1}{x}}$$

Now, if we take the natural log of both sides we get,

$$\ln(y) = \ln\left(x^{\frac{1}{x}}\right) = \frac{1}{x}\ln x = \frac{\ln x}{x}$$

Let’s now look at the following limit and use L′Hopital’s Rule on the ∞/∞.

$$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

This limit was just a L'Hôpital's Rule problem and we know how to do those. So, what did this have to do with our limit? Well first notice that,

$$\boldsymbol{e^{\ln(y)} = y}$$

and so, our limit could be written as,

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} y = \lim_{x\to\infty} \mathbf{e}^{\ln(y)}$$

We can now use the limit above to finish this problem.

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} y = \lim_{x\to\infty} \mathbf{e}^{\ln(y)} = \mathbf{e}^{\lim\limits_{x\to\infty} \ln(y)} = \mathbf{e}^{0} = 1$$

So, there you have it. With L'Hôpital's Rule we are now able to take the limit of a wide variety of indeterminate forms that we were unable to deal with prior to this section.

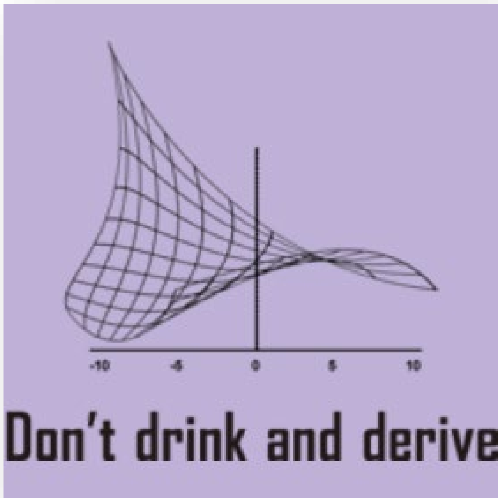
Wrap Up

Okay, that's all I have for today, why don't you march over to WebAssign and get started?

Enjoy your day!

"It has often been observed, that those who have the most time at their disposal profit by it the least. A single hour a day, steadily given to the study of some interesting subject, brings unexpected accumulations of knowledge."

William Ellery Channing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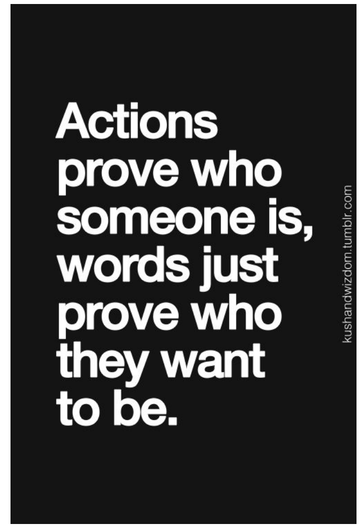
Actions prove who someone is, words just prove who they want to be.
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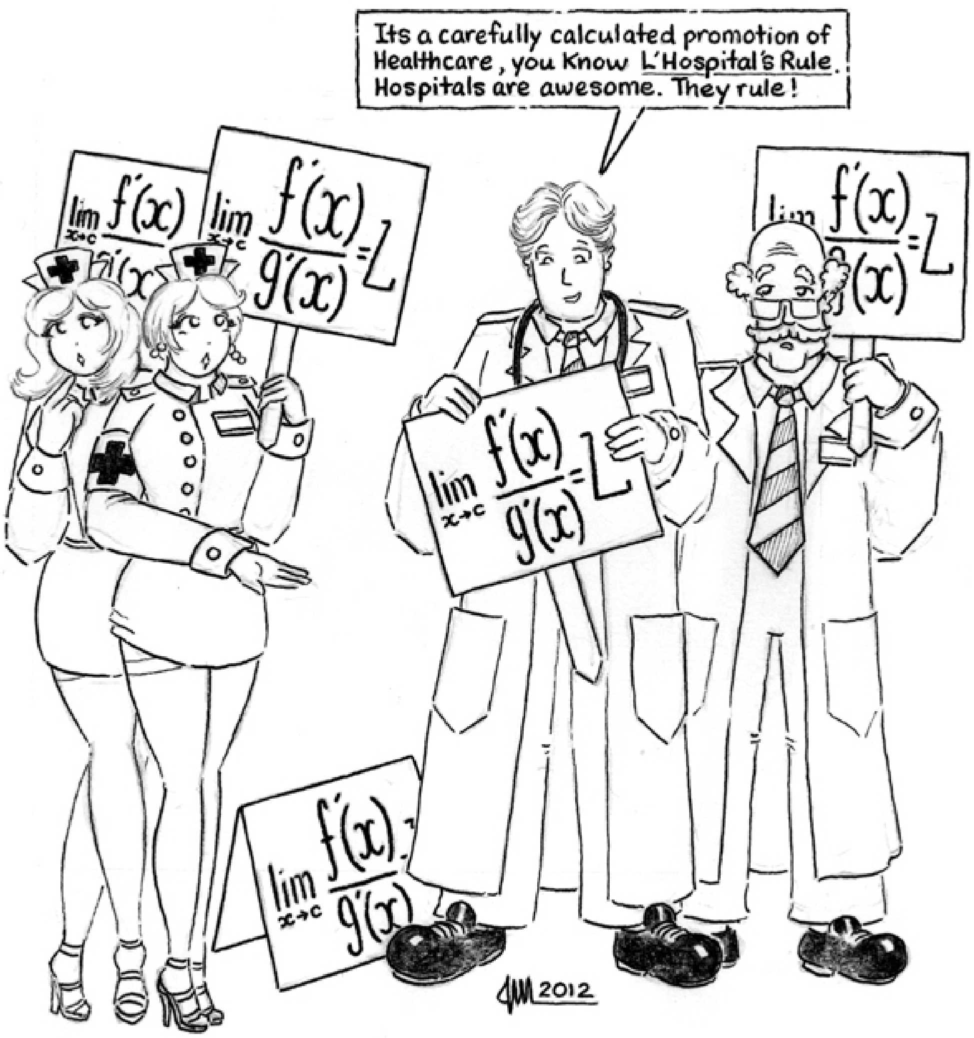
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