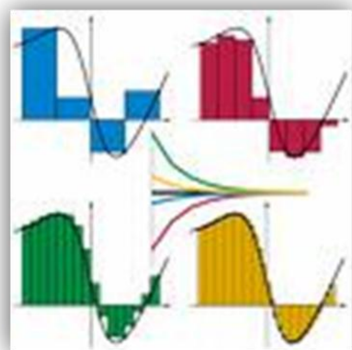
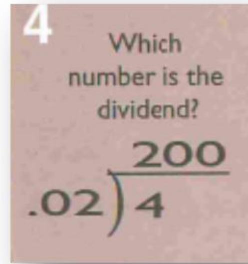
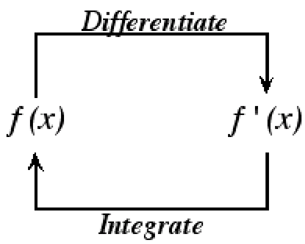




Good morning! Boy these Fridays can sneak up on you when you're not looking, can't they? I was sure today was Thursday. Oh well, I guess I'll just have to settle for the end of the week. I can get happy about that with no effort at all.

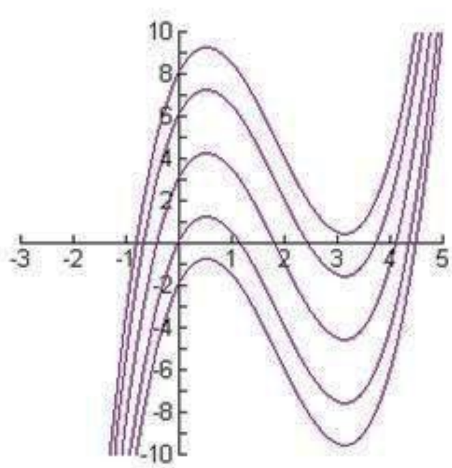
Relationship Between Differentiation and Integration

I hope that you're clicking with indefinite integrals. Sometimes a picture is worth a thousand words. Here's a rather simplistic view of the relationship between differentiation and integration:



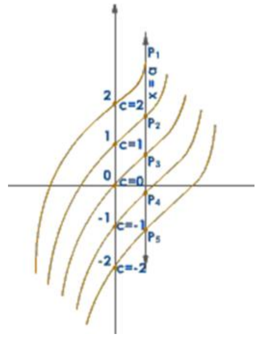
The Initial Value Problem

Here are some families of antiderivatives where they just differ by a constant C:



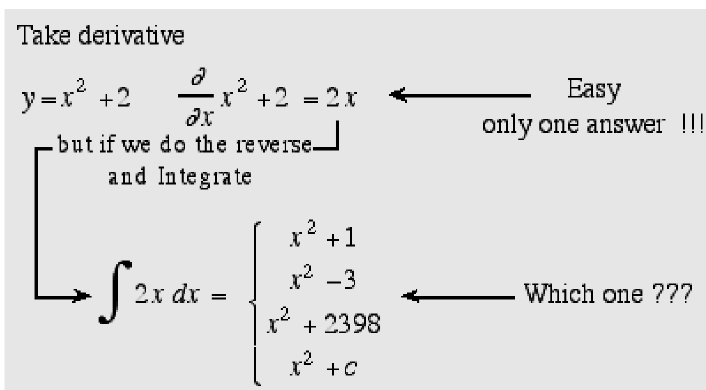
Let $f(x) = 3x^2$

$$\Rightarrow \int f(x)\,dx = x^3 + C$$



You might be asking at this point, "What good does it do to take the integral of a function if all you get is an infinite number of solutions?" One of the great things you can do with calculus is that you can get back to a single function when you know its derivative coupled with a single "initial" value.

By differentiating a function you are determining its rate of change. Because the original function in differentiation has the constant already attached to it, which is zero or an actual number, the solution is unique. On the other hand, if you go backwards and are given a function that represents a derivative of another function, that function can have many solutions, or antiderivatives.



If you're given a value (called an initial value), then you'll be able to pick out of all the antiderivatives the correct one. The initial value or condition is a point on the curve which will isolate the particular curve you want out of all of the curves in the family. That makes sense, doesn't it? This is what we call an Initial Value Problem.

From a Tangent to a Curve to Area Under a Curve - What a Leap!

Let's move on to Section 4.2 – Area. This section is an introduction to the main application of integrals, namely, finding the area under a curve.

Of all things, who would have thought that you could go from a tangent to a curve, to a derivative, to an antiderivative, to the area underneath a closed interval of that curve? Well, it wasn't obvious to the discoverers, Newton and Leibniz, either. Prior to the discovery of the connection between the two methods of calculation, mathematicians had only an inkling that differentiation and integration were related.

It wasn't until the proof of the Fundamental Theorem of Calculus that the connection was made "official." I guess I just let it out of the bag – the biggest event of the first year of studying calculus – (drum roll) – the Fundamental Theorem of Calculus. It is the most important and the most fundamental of all of the theorems of calculus, for the very reason that I just mentioned. It establishes the connection between differentiation and integration. It's coming up soon, so you won't have to wait long.

Lots of New Ideas and New Terms - Let's Get the Big Picture First

Today you are going to read a section that's filled with new ideas, new terminology and a lot of *i*'s and *n*'s. So what I'm going to do is give you the big picture from 50,000 feet before you get mired in the detail. I'm going to appeal to your sense of intuition and your sense of practicality.

Picture This

Let me begin with your intuition. Picture a graph of a function with two endpoints marking off a closed interval [a,b]. Then drop two vertical lines down from those endpoints to the *x*-axis. You should see an area bounded by the curve on top, the vertical lines on the sides, and the *x*-axis on the bottom. The area that the integral equals is the area we just marked off.

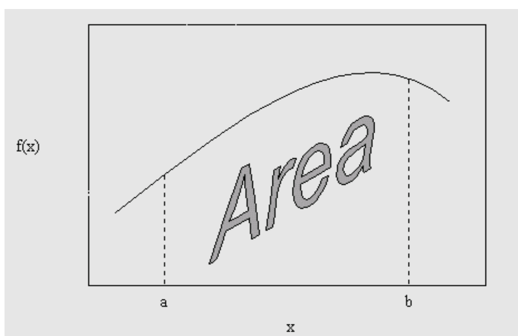


Figure 1

How Can We Approximate the Area without Calculus?

So how would you go about representing the calculation of the area underneath this curve we just drew? (Figure 1) Without calculus, about all we can do is approximate the area by drawing a rectangle with one side from *a* to *b* and the other side the height at *b*. The approximate area would be the length times the height. In this case the length would be (b - a) and the height would be $f(b)$, right?

Now take a look at the slanted line below. With a slanted line, drawing a rectangle to approximate the area would really overestimate the area. In the first graph you can see we dropped two lines down at *a* and *b* then cut the slanted line in half and formed two rectangles instead of just one. Then we could add the areas of the two new rectangles together. You would have a better approximation of the area under the slanted line with two rectangles than with one, wouldn't you?

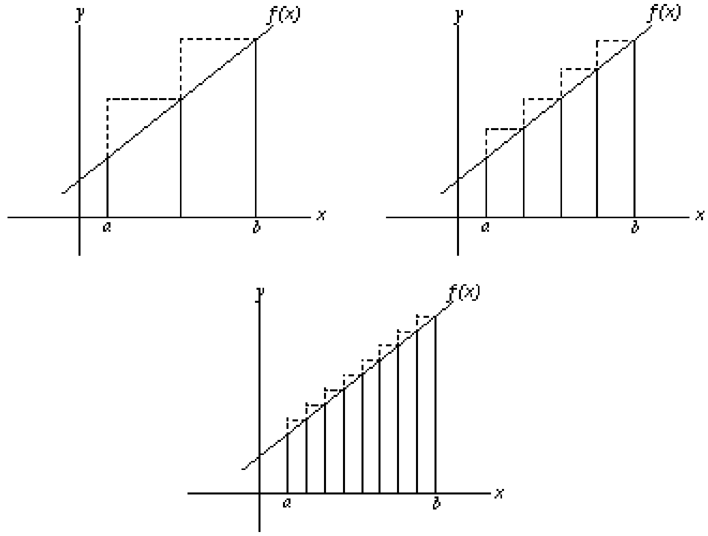


Figure 2

By the way, looking at the first graph above, what's the width of each half? $\frac{b-a}{2}$ Right? And the height of the first rectangle is … well, there's something missing. Let's call the point in between *a* and *b*, *c*, then the height would be *f* (*c*). So you can see how we could calculate the area of each rectangle and add them together to approximate the area under the slanted line.

Here Comes the Intuitive Part

I know, you're thinking, I'm feeling pretty mired, when is she going to get to the intuition. Well here it is. Can you see by looking at the second and third graphs above, if we kept cutting the rectangles under the curve in halves, over and over, forming thinner and thinner rectangles, the sum of their areas would get closer and closer and closer to the area under the curve? I have a hunch that your intuition tells you it would.

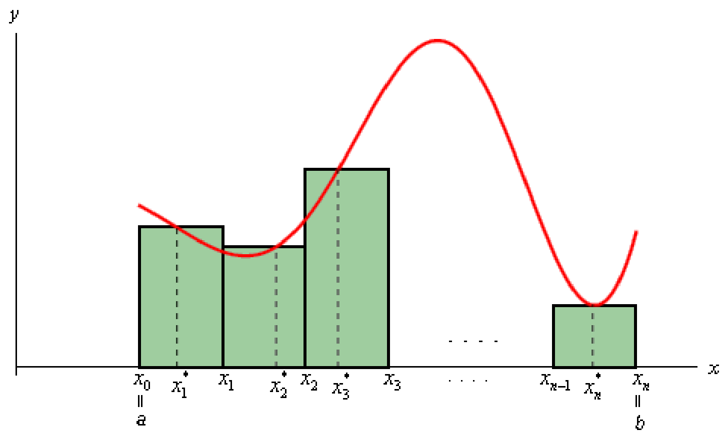
Here is where our old friends "limits" and "infinity" come on the scene. Can you guess why? Well, if you take the limit, as *n* goes to infinity, of the sum of the areas of *n* rectangles with heights [some representation of the heights of the rectangles - *f* (*c*) something] times [some representation of the widths of the rectangles] then you will end up with the area underneath the curve, which ends up being called the integral of the function.

Let's Get Technical with All of the *i*'s and *n*'s

Let's go back to how we represent the heights and widths of the rectangles. As I mentioned when we were cutting the area under the curve in Figure 2 in half, we needed to pick a name for the point in between *a* and *b*, so I picked *c*. Well, I didn't pick *c*. The people that run the show picked *c*. So we represent the heights as $f\,(c_i)$ and say $i = 1$ to *n* (we use *n* to represent the number of rectangles) – remember *n* goes to infinity.

If you remember the width was b - a to start with, then it was $\frac{b-a}{2}$. Well, that handled cutting it in half but cutting it in smaller and smaller pieces gets harder to represent. In fact it's downright messy. It involves a lot of x sub i's and x sub n's. It's not really that hard to understand, it's just cumbersome.

There's a happy ending though. We end up calling the width Δx. Isn't that great? Just plain old delta x. The change in x or the difference between x_2 and x_1 is just Δx. Don't you just love it?



The area under the curve on the given interval is then approximately the sum of the areas of the rectangles, which is simply the individual lengths ($f(x)$) times the widths (Δx).

$$A \approx f\left(x_1^*\right)\Delta x + f\left(x_2^*\right)\Delta x + \cdots + f\left(x_i^*\right)\Delta x + \cdots + f\left(x_n^*\right)\Delta x$$

Here Comes the Practical Part

Now to your sense of practicality. We need to represent the sum of all those rectangles somehow. This is where the Greek letter for capital S $\sum$ comes in. It's shorthand for "the sum of." Then we put $i = 1$, or whatever the starting point of the summation is, at the bottom of the sigma notation and n at the top. We call "i" the index of summation.

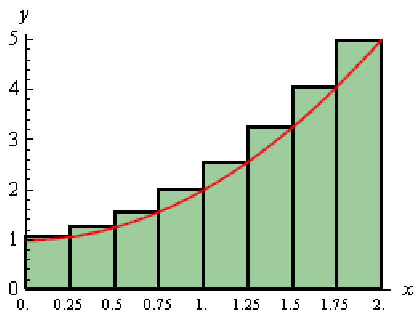
It's just a matter of practicality, and a way for mathematicians to use more symbols. It's a method of writing a very long cumbersome summing of numbers in a very succinct, compact format. Once you get used to the notation, and it does take some getting used to, you'll see how much better it is than having to write out the sum of all those numbers in long hand.

$$Area \approx \sum_{i=1}^{N} f_i \Delta x$$

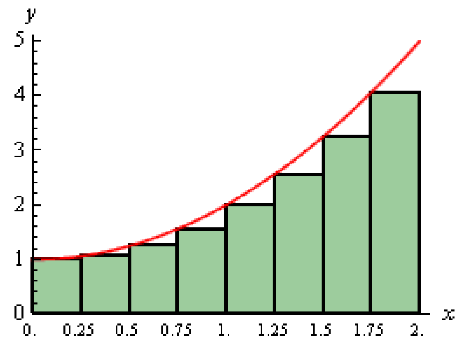
There are Different Kinds of Rectangles

Now that I've told you about adding up the rectangles and making them smaller and smaller to get better and better approximations of the true area under the curve, I'm going to complicate it, just a little, by telling you, there are different kinds of rectangles. You can draw the rectangles with equal subintervals, Δx's, or they can be unequal. For certain purposes you can draw them with what are called right side endpoints, left side endpoints or midpoint endpoints.

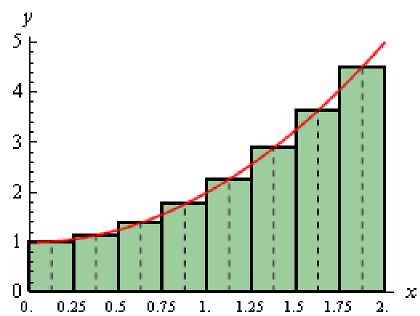
You can see some rectangles pictured below that use these types of endpoints. Pay attention to the difference between areas calculated using left, right and mid-point sums. You may be asked to find one or the other on the AP Exam. Let's use the following examples to approximate the area under the given curve. The numbers along the *x*-axis are in increments of 0.25. They're probably too small for you to see.



Right-Hand Endpoints



Left-Hand Endpoints



Mid-Point Endpoints

Here are the area estimations for each of the above sums.

$$A_r = 5.1875 \qquad A_l = 4.1875 \qquad A_m = 4.65625$$

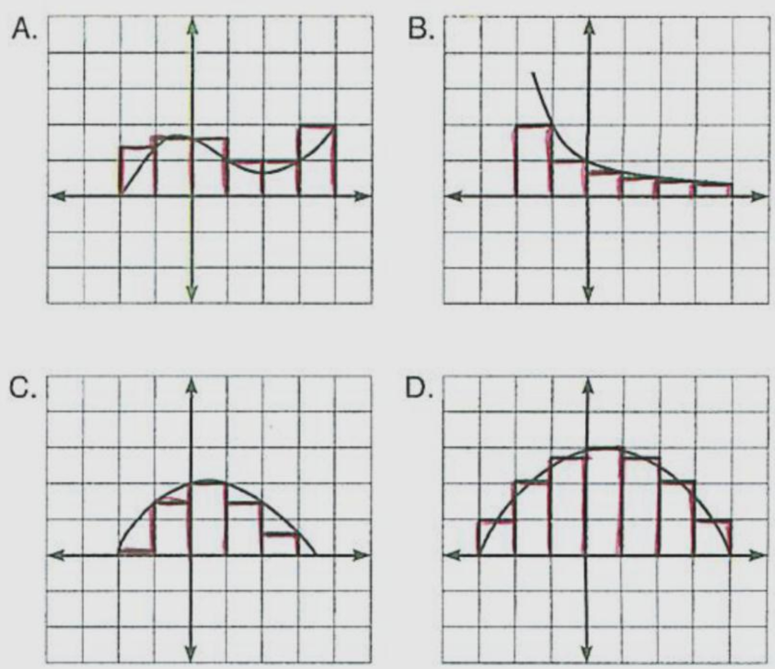
The midpoint sum, on the far right, is the closest approximation to the actual area under the curve. Can you tell by looking at the graphs that the midpoint sum would give you a closer approximation than the right or left endpoint sum? It is usually better to use a midpoint sum to approximate the area under a curve. Intuition should tell you this. Intuition is usually what leads people down the path to discovering proofs of theorems, such as Newton and Leibniz discovering the Fundamental Theorem of Calculus. See how I slipped that in again, pretty clever, huh?

Upper Sums from Circumscribed Rectangles and Lower Sums from Inscribed Rectangles

You may also hear a reference to upper and lower sums. An upper sum is a sum that is the result of all "circumscribed" rectangles, which are rectangles that lie outside the individual intervals (see the right-hand sum above). A lower sum is the result of the sum of the areas of "inscribed" rectangles, which are rectangles laying inside the individual intervals (see the left-hand sum above).

I know, you feel really mired in details now. That's a lot of terminology to throw at you all at once. Here's a little test for you to see if you got all of what I just said. The answers are on the next page, so you can check to see if you got them right. Try to see if you can tell what kind of sums these are: upper, lower, or midpoint sums? Right, left, or midpoint rectangles? Circumscribed or inscribed rectangles?

What type of Riemann sum is being applied in each of the following diagrams? If there is more than one correct answer, give both.



(a) Upper sums, circumscribd rectangles: On the first two rectangles, the right-hand endpoint is used to determine height, whereas the left-hand endpoint is being used for the third and fourth rectangles. Thus, it cannot be right- or left-hand sums.

(b) Lower sums, right-hand sums, inscribed rectangles: All these descriptions apply to this diagram since the right-hand endpoint of each interval forms the inscribed rectangles.

(c)Lower sums, inscribed rectangles: The lower of the two endpoints' heights is chosen each time, not the right- or left-hand endpoint on a consistent basis.

(d) Midpoint sums: That one's pretty clear from the diagram. The function value at each interval midpoint dictates the height of the rectangle there.

I hope this introduction will make your reading today easier to understand and visualize.

We're Talking About a Subset of Functions

Whew! That was a lot of new material. Now, you should note that this region is a restrictive region. It is bounded by the *x*-axis on the bottom, two vertical lines where $x = a$ and $x = b$ on the sides, and the curve on the top. Because we are talking about area, the region is ***positive***.

So Section 4.2 is definitely not talking about all curves, or in other words, all functions. The purpose of this section is to get you familiar with some new terms, using a subset of curves, so that they only need to deal with a subset of the new terms.

Wrap Up

That's all I have for today, why don't you go to the Current Assignments page and get started with today's assignment.

Have a great day!