Leonhard Euler (1707-1783)

AP Calculus AB
Unit 5.1
Day 90

Good morning!

Unit: 5.1 – Slope Fields and Euler's Method

Today we start Chapter 5. The first section is on Slope Fields and Euler's Method. So, who was Leonhard Euler (pronounced *Oiler*)? Here's what Wikipedia said about him:

> "Euler made important discoveries in fields as diverse as calculus and graph theory. He also introduced much of the modern mathematical terminology and notation, particularly for mathematical analysis, such as the notion of a mathematical function.
>
> Euler is considered to be the preeminent mathematician of the 18th century and one of the greatest of all time."

Wow! Now that's what I call quite a biography – "the preeminent mathematician of the 18^{th} century and one of the greatest of all time!" This guy must have been really, really, smart! Whew!

So how does he figure in today's lesson? Euler's Method is a numerical approach to approximating the particular solution of a differential equation, and today is all about differential equations.

So What Exactly is a Differential Equation?

Let's back up and review for a minute. What is a differential equation? It's nothing more than an equation containing a derivative. We have created many differential equations simply by finding derivatives of functions.

In this chapter, we'll start with the differential equation $f'(x)$ (the derivative) and work backwards to find the original equation $f(x)$ its antiderivative. Now if you stop and think about it, all we'll be doing is applying integration methods. It should intuitively make sense that to solve a differential equation, you would integrate it.

How We Got Where We're At

In the beginning of the book, when we were talking about derivatives, and solving for derivatives, we were given equations that looked like this:

$$s(t) = t^2 + 3t + 2$$

And we were asked to find the derivative, which would be:

$$s'(t) = 2t + 3$$

Then we learned about going backwards using integration, so we were given equations like this:

$$s'(t) = 2t + 3$$

And we were asked to find the antiderivative (indefinite integral), which would be:

$$\int 2t + 3\ dt = 2\int t\ dt + \int 3\ dt = 2\frac{t^2}{2} + 3t = t^2 + 3t + \text{C}$$

And if we wanted a particular solution at an initial value or point, we would solve for C at that point like this:

Find particular solution at (0, 2):

$$\begin{aligned} &y = t^2 + 3t + C \quad \text{at} \quad (0, 2) \\ &2 = 0 + 0 + C \\ &2 = C \\ &y = t^2 + 3t + 2 \end{aligned}$$

Modeling Real Life Brings the Function and Its Derivative Together in One Equation

In real life, things are not quite so simple. It turns out, when you model real-life phenomena, in other words, real-life situations, you end up with equations that contain both an unknown function, such as $f(x)$, and its derivative $f'(x)$, such as:

$$f'(x) = x - f(\text{x}) \qquad \text{another way of writing this is} \qquad y' = x - y$$

These are both saying the same thing and they are both differential equations, because they have derivatives in them. This is a logical topic to follow differentiation and integration, because it involves both techniques. It was necessary for you to learn each technique separately, and it is logical that to model some things, you need both in the same equation.

Solving Differential Equations Results in a Family of Solutions

You remember that finding antiderivatives, or indefinite integrals, resulted in a family of solutions, so we had to always add a constant at the end of a solution. It follows that you need to do the same thing with differential equations, because you are normally solving for the original function, $f(x)$ or y in the second format, which is its antiderivative, so you need the constant.

Because in real life you're typically looking for a particular function, and you know a particular *x* and *y* value of the function you're looking for, you can plug in the specific values into the general solution, then solve for *C*, then plug *C* into the solution, and viola´, you've got the particular function you were trying to find.

Some Differential Equations Can't Be Solved or Are Very Difficult to Solve

Just like integrals, there are a lot of differential equations that cannot be solved or are very difficult to solve using just equations, or what's called analytically. So, some very smart mathematicians thought of a way to approach this problem graphically.

Obviously, if you could draw the graph of the functions in the family of solutions, then you would know the shape of the graph of the particular solution function you were looking for. But you can't do that because you can't solve the equation.

Sketching a Particular Solution Using a Slope Field

So, this is where the graphs come into play. Here's what they figured out: if you found the slope at particular points and drew short segments at those points, and you did this in all four quadrants of the *xy* plane of a graph, then you could get a pretty good idea of the direction of the family of solutions. Then you could take the particular values you had for *x* and *y*, and find the particular curve that point fell on, and that curve would be your particular solution curve.

Now, I'm saying "curve" in a loose sort of way, because it's not a curve, it's a bunch of small segments. But you can get a pretty good idea of the direction of the curve. If you put all these short segments from the family of solutions together, you have what is called a "slope field."

You can find the slope of these short segments because you know the derivative, cuz what's the derivative? It's the slope. I know, you already knew that. I'm just saying it because it's so neat that they figured out how to use this idea, those very smart mathematicians I just mentioned.

So, what does a slope field look like? Here are some examples showing a slope field with particular solutions drawn in on top of them:

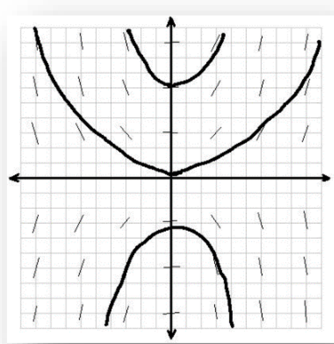


Figure 1. Slope Field with Particular Solutions

The short segments in the graph represent the slopes at points of the family of solutions to the given differential equation (whatever it is). The curves drawn in are the particular solutions amongst the family of solutions.

Here's another example:

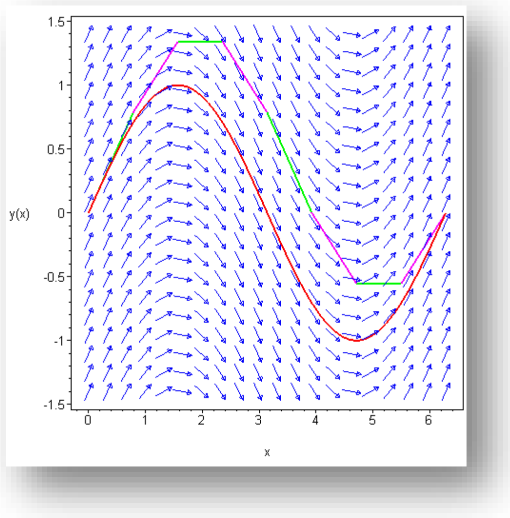


Figure 2. Example of a Slope Field

Examples of Slope Fields with Particular Solutions

The general differential equation $y' = f(x, y)$ gives you a slope at each point in the (x, y)-plane. If we take the equation $y' = xy$, we can plot the slope at lots of points as a short line through the point (x, y) with the slope y'. See Figure 3.

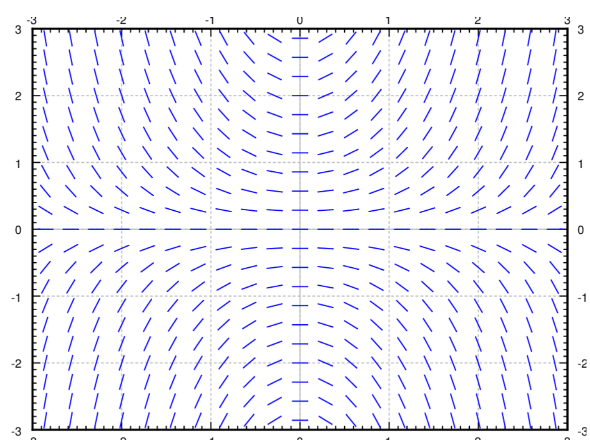


Figure 3. Slope field of $y' = xy$.

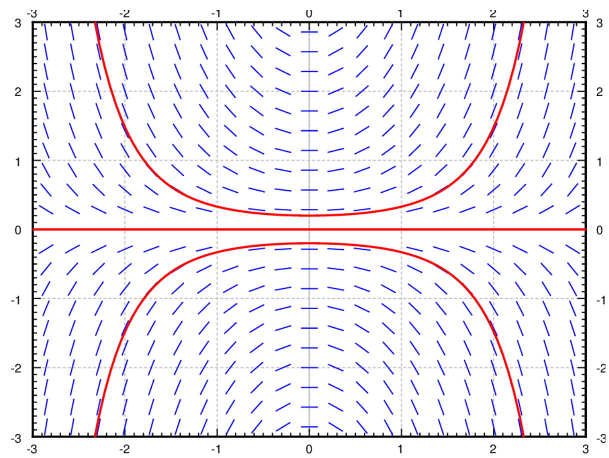


Figure 4. Slope field of $y' = xy$ with a graph of solutions satisfying $y(0) = 0.2$, $y(0) = 0$, and $y(0) = -0.2$.

By looking at the slope field we can get a lot of information about the behavior of solutions. For example, in Figure 4 we can see what the solutions do when the initial conditions are $y(0) > 0$, $y(0) = 0$ and $y(0) < 0$. Note that a small change (± 0.2) in the initial condition causes quite different behavior.

On the other hand, plotting a few solutions of the equation $y' = -y$ in Figure 5, we see that no matter what $y(0)$ is, all solutions tend to zero as x tends to infinity. See Figure 5.

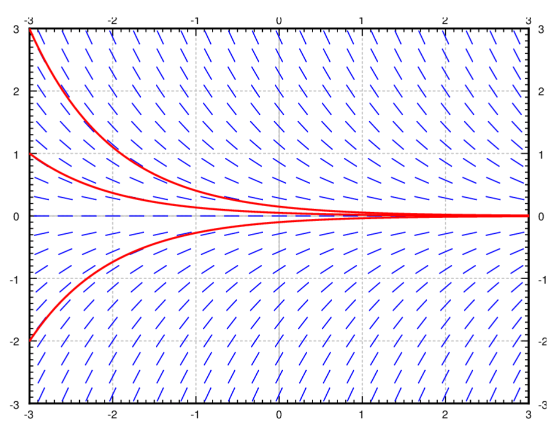


Figure 5. Slope field of $y' = -y$ with a graph of a few solutions.

Euler's method

Euler's Method will not be on the AP Calculus AB Exam, but I'm having you read about it for three reasons:

1) he's a very important person in the history of mathematics,

2) you'll study it in the next course of college calculus if you take it in college, and

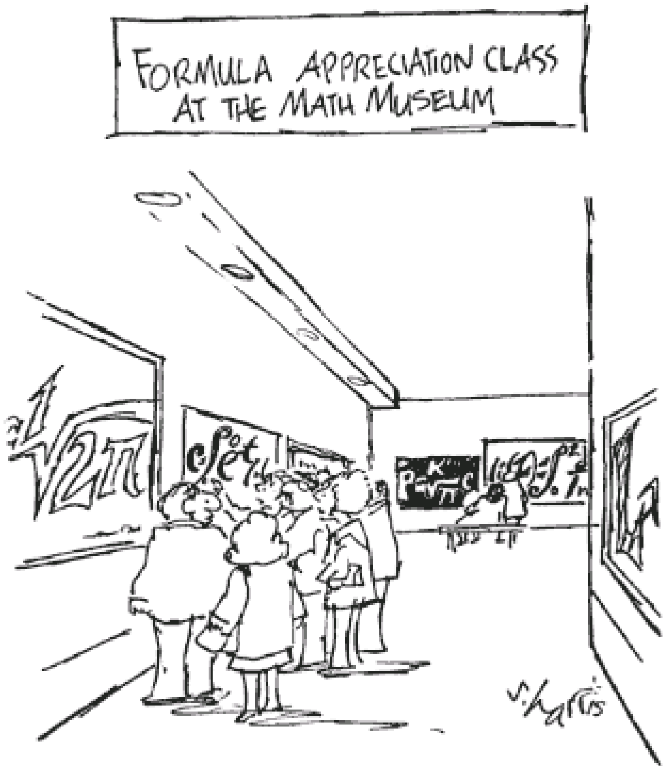
3) it’s not that hard of a subject. But if you don’t understand it, you won’t have to worry about it for the Exam.

Wrap Up

That’s all I have for today. It’s time for you to slide over to WebAssign and get started on your assignment to see how well you understand this new and exciting (I hope) application of what you’ve studied so hard to learn, differentiation and integration.

Enjoy your day!!

"*Our lives begin to end the day we become silent about things that matter.*" — Martin Luther King, Jr.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