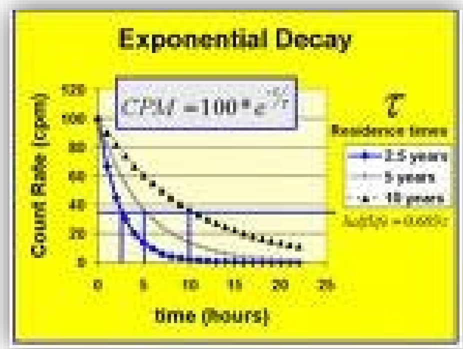


AP Calculus AB
Unit 5.2
Day 92

Good morning!

Review of Slope Fields

Let's get on with Chapter 5. Let's see, where were we? Ah yes, Differential Equations – Slope Fields and Euler's Method. Here's a quick review of slope fields:

Slope Fields – Today's Example #1

Slope fields (also called **vector fields** or **direction fields**) are a tool to graphically obtain the solutions to a first order differential equation. Consider the following Example #1:

$$y' = -2xy \, . \qquad \text{Example \#1}$$

We can determine a slope, *y* '(*x*), of one of the solutions *y* (*x*), by selecting actual values for *x* and *y*. For example, if I select *x* = 1 and *y* = -1, then the slope of the solution *y* (*x*) passing through the point (1, -1) will be:

$$\begin{aligned} y'(x) &= -2xy \\ &= (-2)\cdot 1\cdot(-1) \\ &= 2 \end{aligned}$$

If we graph *y* (*x*) in the *x*-*y* plane, it will have slope = 2, given *x* = 1 and *y* = -1. We show this graphically by inserting a small line segment at the point (1, -1) of slope 2.

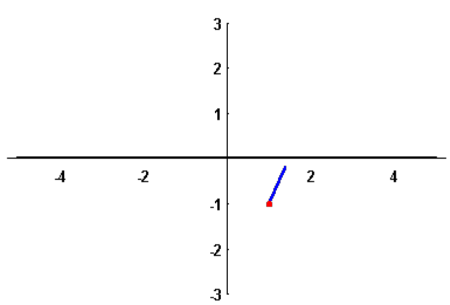


Figure 1. Graph of one solution slope = 2 at (1, -1) for differential equation $y' = -2xy$.

Therefore, the solution of the differential equation with the initial condition $y(1) = -1$ will look similar to this line segment as long as we stay close to $x = 1$. Of course, doing this at just one point does not give much information about the solutions. We want to do this simultaneously at many points in the *x*-*y* plane as in Figure 2.

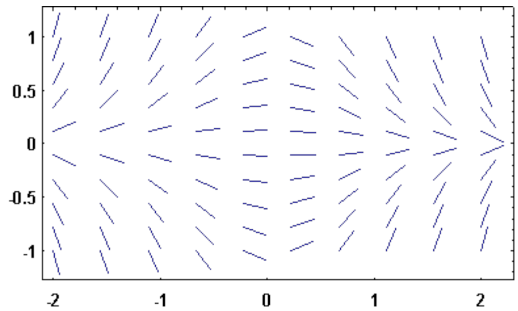


Figure 2. Graph of many solutions' slope segments for differential equation $y' = -2xy$.

We can get an idea as to the form of the differential equation's solutions by "connecting the dots." So far, we have graphed little pieces of the tangent lines of our solutions. The "true" solutions should not differ very much from those tangent line pieces as shown in Figure 3 where several solution curves have been drawn:

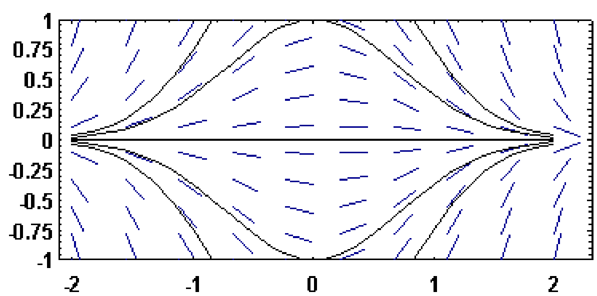


Figure 3. Graph of 2 particular solutions on the slope field for differential equation $y' = -2xy$.

That's a summary of what you learned about slope fields.

A General Solution to a Differential Equation – Today's Example #2

Then we looked at a differential equation like Example #2:

$$xy' - 3y = 0 \qquad \text{Equation [1] – Differential Equation}$$

and we were told that

$$y = Cx^3 \qquad \text{Equation [2] – General Solution to Equation [1]}$$

was a general solution for that equation, which we could check by substituting y and y' back into Equation [1] and see if it equals 0.

$$
\begin{aligned}
xy' - 3y &= 0 \\
&= x(3Cx^2) - 3(Cx^3) \\
&= 0
\end{aligned}
$$

Equation [3] – Verifying [2] is General Solution to [1]

It worked! Did you follow that? I just plugged in what they told us was the general solution in Equation [2], y (Cx^3) and its derivative y'($3Cx^3$), into the original Differential Equation [1] to see if I would get 0, and I did. So, we verified it is a solution.

Equation [2] is a general solution because it's got that C, which stands for the constant of integration, which we always use when we solve by going backwards from a derivative to the original function, its antiderivative, by integrating the differential equation.

A Particular Solution to a Differential Equation using today's Example #2

Then they gave us what they called an initial condition like y = 2 and x = -3. Given those two values for the initial condition, you were asked to find a particular solution. To do that you substitute those two values for x and y in the general solution and solve for C under those conditions. Let's do that:

$$
\begin{aligned}
y &= Cx^3 \\
2 &= C(-3)^3 \\
-\frac{2}{27} &= C
\end{aligned}
$$

So, that's what C equals when x and y equal 2 and -3 respectively. Then you take the general solution and put in that value in place of C, and then you've got your particular solution. This is it:

$$
y = -\frac{2x^3}{27}
$$

Matching a Particular Solution Back to the Differential Equation's Slope Field using Today's Example #1

Now that we know how to find a particular solution let's see if we can find a particular solution that will match our slope field from Example #1. Let's get the **general solution** first:

$$y' = -2xy$$
$$\frac{y'}{y} = -2x$$
$$\int \frac{1}{y} dy = -2\int x dx$$
$$\ln|y| = -2\left(\frac{x^2}{2}\right) + C$$
$$\ln|y| = -x^2 + C$$
$$|y| = e^{-x^2+C}$$
$$|y| = Ce^{-x^2}$$
$$y = C\frac{1}{e^{x^2}}$$

So, we have the general solution. Let's pick an initial condition (1,1), solve for C and get a **particular solution**:

$$y = C\frac{1}{e^{x^2}}$$
Let $x = 1$ and $y = 1$
$$1 = C\frac{1}{e^1}$$
$$e = C$$
Substitute C in general solution
to get particular solution:
$$y = \frac{e}{e^{x^2}}$$
$$y = e \cdot e^{-x^2}$$
$$y = e^{1-x^2}$$

Okay, let's graph that particular solution to see if it matches our **slope field** in Figure 2 and Figure 3:

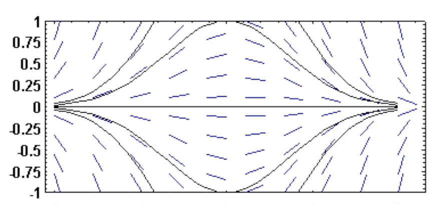


Figure 3. Graph of 2 particular solutions on the slope field for differential equation $y' = -2xy$.

Here's a graph of our particular solution that goes through (1,1). See if it fits the slope field in Figure 3.

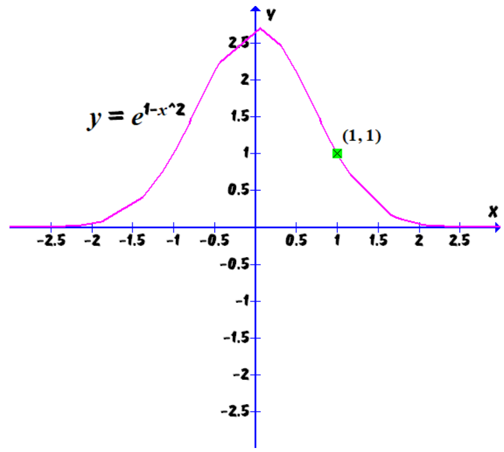


Figure 4. Graph of our particular solution with initial condition x = 1, y = 1 to be matched to the slope field for differential equation $y' = -2xy$ in Figure 3.

There you have it. The function that we calculated to be a particular solution to the given differential equation with an initial condition of (1, 1) fits very nicely with the slope fields in Figure 2 and Figure 3.

You can check the validity of these concepts we are learning as we go along, to make sure they actually work, any time you want to. It can help you understand and visualize the concepts we are studying, by tying them together as we move through the sections.

We just tied together the graphs of slope fields, to the calculation of a general solution, and then to the calculation of a particular solution. I got into material that we will be covering in more depth in sections 5.2 and 5.3, so stay tuned for more information and background on what we just did.

Example #5 from Section 5.1

I would like to clear up two questions I received. One of them was about the reading from Section 5.1. Someone asked a very good question about Example #5 given in the book, where they sketched a slope field. They showed a table with values for x and y and another row for $y' = 2x + y$. It was clear that they had picked arbitrary values for x from -2 up to +2 to start the table off. The question was "Where did the y values come from?" How were they calculated?

Well, those y values were arbitrary too. They could have been any values. Once they picked a pair for x and y, then they solved for y′ by plugging in those values for x and y. They're just trying to find what the slopes would look like at various places on the graph, so they can get a general idea of what the slope field would look like at those points. For example:

I could have picked x = -3 and 3 with corresponding y = 2 and 2. I just picked those four numbers out of a hat. Then $y' = 2x + y$ would be -4 and 8 respectively.

So, then I'd go to the graph and go to those points (-3,2) and (3,2) and plot two short lines:

- ✓ one at (-3,2) with a slope of -4 (backwards with a slightly steep slope)
- ✓ and one at (3,2) with a slope of 8 (forwards with a really steep slope) respectively.

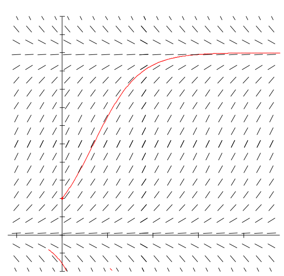
Homework Question #69 from the Larson Textbook from Section 5.1

The second question was about #69 from Section 5.1 where I asked you to use the computer algebra system WinPlot or Mac's Grapher.

To get the slope field and particular solution for question #69 using WinPlot you need to follow these steps:

1. Click on [Window] and then [2-dim] in the small WinPlot window
2. Click on [Equa] [Differential] [*dy*/*dx*]
3. Enter 0.02y(10-y) next to *dy*/*dx*=
   (that's the equation for problem #69 on page 405 from Section 6.1)
4. Click on [View] [View…] left = -12, right = 48, down = -2, up = 12
   (those are the dimensions for the window)
5. Click on [One] [*dy*/*dx* trajectory]
6. Enter 0 next to x = and 2 next to y = click on [Euler] click on [draw]

If I did everything right, you should see the following graph:



$dy/dx = 0.02y(10-y), \quad y(0) = 2$

Unit 5.2: Growth and Decay

Let's look at Section 5.2 in our textbook. Growth and Decay. Knowing how to solve a differential equation opens up the possibility of solving problems where you know the rate of change, or derivative, of a function and you want to work backwards to find the original function, its antiderivative.

Are There Any Techniques for Solving Differential Equations?

Before we get to a real-life application of what we're learning let's stop and ask a very practical question: Are there any techniques for solving a differential equation? I mean, there are all these x's and y's and y primes and y double primes. Geez, whatever happened to adding 2 to both sides, dividing by 4, and solving for x?

You remember Algebra, where you spent all that time learning how to solve equations that ended up with x and y equal to a number. Well now we're solving differential equations with derivatives as unknowns and solving for functions of x instead of just x. We're ending up with an equation with a constant C in it, as a general solution.

Then if we want to, which we usually always do, we can select an initial condition and substitute it into the general solution to solve for that constant C, and that's when we get a particular solution which is just a plain old equation with an x and y as unknowns.

Don't you sometimes wish for the good old days? Well, don't give up yet. There are different ways of solving differential equations. They're definitely not as neat and clean as the ones you're used to. But the ones we'll be dealing with here will be solvable (well that's a relief), and we're about to learn a technique.

Separation of Variables – A Technique for Solving Differential Equations

The technique is called "separation of variables." It's a fancy name for rewriting the equation with all the like variables on one side of the equation, like this:

$$y' = \frac{\sqrt{x}}{3y}$$

$$3yy' = \sqrt{x}$$

I just multiplied both sides by $3y$ so I had all the y's and y primes on one side and the x's on the other. Pretty simple, huh? That's separation of variables in a nutshell. So now, how do we solve it? It looks pretty messy, doesn't it? And what are we solving for? Our ultimate goal is usually a particular function or curve that represents the same relationship between x and y but doesn't have any derivatives in it. But we've got to solve for the general solution to that differential equation that we started off with first.

Once I know the general solution, then if I know a particular x and y, I can single out which one of those many, many curves is the one that solves my problem. It will tell me the curve I'm looking for. That's usually the ultimate goal.

So, we're trying to first find the general solution to the differential equation. Once we've got all the variables on their own sides, what do we do then? If we want to get rid of the derivatives and go backwards to the original function, what do we do? We integrate! So, that's what we'll do:

$$\frac{dy}{dx} = \frac{\sqrt{x}}{3y}$$
$$3y\ dy = \sqrt{x}\ dx$$
$$\int 3y\ dy = \int \sqrt{x}\ dx$$
$$\frac{3y^2}{2} = \frac{2}{3}x^{\frac{3}{2}} + C_1$$
$$9y^2 - 4x^{\frac{3}{2}} = C$$

Notice I changed to *dy*/*dx* for the derivative. I did that to show you how the *dy* and *dx* get into the integrand. Make sure you followed every step. It's a simple integration problem. Our end result is a family of curves, or functions, that solve the original differential equation we started with. If we had a point that was on the particular function we were looking for, then we could solve for the actual function that would be our particular solution.

How Would We Use Differential Equations in Real Life?

So, what kinds of real-life applications use differential equations? Have you ever heard someone say, "At this rate I'll never have enough money to pay for that car?" or "At this rate the population of the earth will reach 7 billion in the year xxxx?" (or should I say 7.1?) or "At this rate I'll never finish reading these Comments?"

What do these real-life situations have in common? These applications might be handy in predicting things. They all involve *rates*, which is a real obvious application to anything having to do with derivatives. They all seem to involve time. What else is common to them? It's subtler. The rate of change is proportional to the amount present.

The Rate of Change of the Unknown is Proportional to Itself

For instance, the rate of growth of a population is a function of how many people you start off with, agree? A city with 100 people is going to grow a lot slower than a city with a million people. Isn't the time it takes you to finish these Comments, not only dependent on the rate of how many words you read per minute, but also on the number of words in the Comments?

How would you write an equation of a rate of change that is proportional to the amount that is changing? Take a look at the following equation:

$$\frac{dy}{dt} = ky$$

Notice I replaced *x* with *t*, because we're dealing with rates that have to do with time. The *k* stands for a constant. It's called a *constant of proportionality*, which makes sense. Let's solve this differential equation for y like we would any other differential equation to see what its solution looks like:

Law of Exponential Change

The differential equation that describes growth is $\frac{dy}{dt} = ky$, where k is the growth constant (if positive) or decay constant (if negative). We can solve this equation by separating the variables.

$\frac{dy}{dt} = ky$	
$\frac{dy}{y} = k\,dt$	separate the variables
$\int \frac{dy}{y} = \int k\,dt$	integrate both sides
$\ln\lvert y\rvert = k\,t + C$	
$\lvert y \rvert = e^{kt + C}$	exponentiate both sides
$\lvert y \rvert = e^{C}\,e^{kt}$	property of exponents
$y = A\,e^{kt}$	Let $A = \pm e^{C}$

Our textbook uses *C* instead of *A* in the solution, like so:

$$y = Ce^{kt}$$

where *k* and *C* are constants. To find whether this function satisfies the above differential equation, we will need to compute its derivative, which we do using the Chain Rule. (We need to use the Chain Rule because the function above has an expression $u = kt$ in the exponent, not just *t* alone.)

$$\frac{dy}{dt} = \frac{d}{dt}\left(Ce^{kt}\right) = C\frac{de^{kt}}{dt} = C\frac{de^{u}}{du}\frac{du}{dt} = Ce^{u}k = k\left(Ce^{kt}\right) = ky$$

Notice that we just showed **the function $y(t) = Ce^{kt}$ will solve the equation**

$$\frac{dy}{dt} = ky$$

by substituting it for *y* and then solving for the derivative.

Let's try an example:

Example: Describe the function $y = \$50 \cdot e^{0.20\,t}$ in words and sketch its graph. Assume that t is measured in years.

Solution: This is an exponential growth function. Its value is \$50 at time 0 and it grows at a rate of 20% per year. In the sketch below, we have shown that the slope of the curve is 0.20 times the height at three different points along the curve.

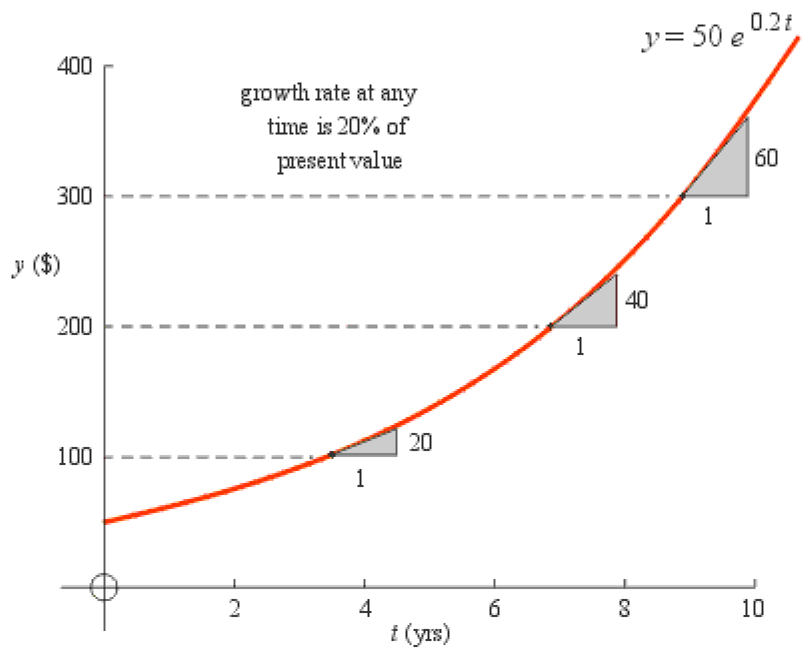


Figure 5. Graph of exponential growth function $y = \$50 \cdot e^{0.20\,t}$

Let's try another example involving compounding interest continuously.

Example: Suppose that we put \$1500 into a bank account which receives interest at a rate of 8% per year and which is compounded continuously. (Compounded continuously is just another way of saying grows exponentially.) Let y denote the amount of money in the account at any time t. Then y can be expressed like so:

$$y = y_0 \cdot e^{r\,t},$$

where $y_0 = \$1500$ is called the principal and $r = 0.08/\text{yr}$ is the interest rate. The units of y will also be dollars and t must be given in years so that the units of r and t cancel. Questions:

(a) What amount will be in the account at the end of 15 months?

(b) After how many years will there be \$4000 in the account?

Solution:

(a) Substituting t = 15 months = 1.25 yrs. into the equation and evaluating gives:

$$y = \$\,1500 \cdot e^{(0.08/yr)(1.25yr)}$$

$$= \$\,1657.75$$

(**b**) Substituting y = \$4000 into the equation and solving for t gives:

$$\$4000 = \$\,1500 \cdot e^{0.08t}$$

$$\ln(\$4000/\$\,1500) = (0.08/yr)\cdot t$$

$$t = 12.26 \text{ yrs.}$$

Answers:

(a) After 15 months the account holds \$1,657.75.

(b) The account holds \$4,000 after 12.26 yrs.

Exponential Decay Toward a Limiting Value

Let's look at an example of exponential decay toward a limiting value. Notice the effect each coefficient has on the shape of the curve.

The figure to the right shows four functions whose *differences* from the limiting value $y = 5$ decay exponentially at various rates. The functions are:

(a) $y = 5 - 5\,e^{-t/2}$
(b) $y = 5 + 3\,e^{-t/2}$
(c) $y = 5 + 3\,e^{-t/6}$
(d) $y = 5 - 7\,e^{-t/6}$

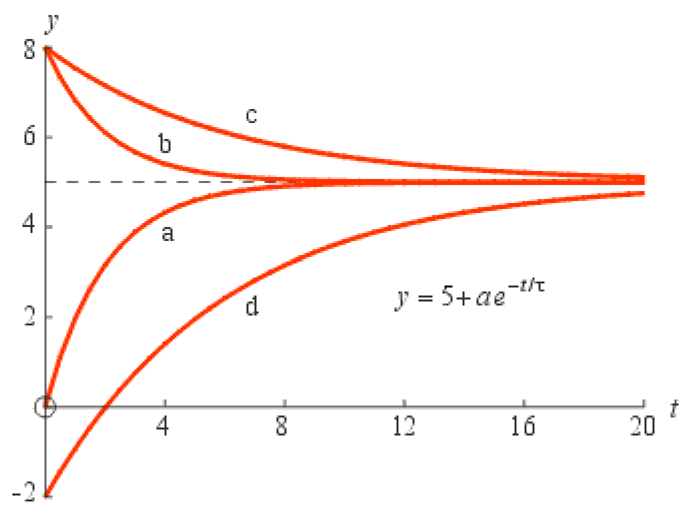


Figure 6. Graph of four functions whose differences from limiting value y = 5 decay exponentially.

These functions could describe the temperature of hot drinks cooling and cold ones warming toward room temperature. (c) and (d) are in better insulated containers so they take longer to heat or cool. These functions are all the form:

$$y = y_\infty + a\, e^{-t/\tau},$$

There are 3 parameters: y_∞, a and τ. If we write transpose y_∞ to the left-hand-side and write the equation as

$$y - y_\infty = a\, e^{-t/\tau},$$

then we see that the right side is the exponential decay in time constant form, and that the difference of y from y_∞ equals a when $t = 0$ and that this difference decays with time constant τ. When $t = \infty$ then $y = y_\infty$. Because there are 3 parameters, we must be given the value of y at 3 different times to fix them.

How Populations Grow Exponentially

Most naturally occurring phenomena **grow continuously**. For example, bacteria will continue to grow over a 24 hours period, producing new bacteria which will also grow. The bacteria do not wait until the end of the 24 hours, and then all reproduce at once.

The **exponential *e*** is used when modeling continuous growth that occurs naturally such as populations, bacteria, radioactive decay, etc. You can think of e like a universal constant representing how fast you **could possibly** grow using a continuous process. And, the beauty of e is that not only is it used to represent continuous growth, but it can also represent growth measured periodically across time.

Example: A strain of bacteria growing on your desktop doubles every 5 minutes. If you start with only one bacterium, how many bacteria could be present at the end of 96 minutes? (*bacteria continuously grow*)

$$A = A_0 e^{kt}$$

Find k first.

$$2 = 1e^{k \cdot 2}$$

$$2 = e^{5k}$$

$$\ln 2 = \ln e^{5k}$$

$$\ln 2 = 5k$$

$$\frac{\ln 2}{5} = k$$

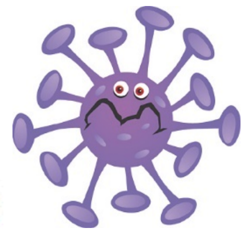
$$k = 0.1386294361$$

Now, form the equation using this *k* value, and solve the problem using the time of 96 minutes.

$$A = A_0 e^{0.1386294361t}$$

$$A = 1 \bullet e^{0.1386294361(96)}$$

$$A = 602{,}248.76225 \text{ bacteria}$$



How populations grow exponentially.

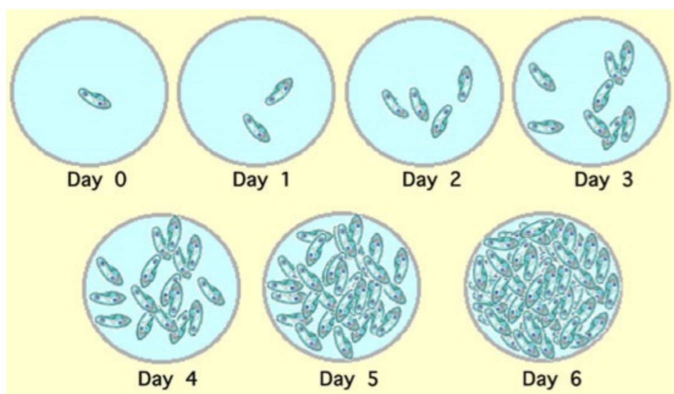


Figure 7. Changes in a population of Paramecium over a six-day period.

Let's Use It to Predict Something

Let's use the Exponential Growth and Decay Model $y(t) = Ce^{kt}$ to predict something. Let's say they know that plutonium decays at a rate proportional to the number of grams of plutonium. So, we know:

Rate of Decay grams/year	=	Constant of Proportionality	x	Grams
$\frac{dy}{dt}$	=	k	*	$y(t)$

They don't know the rate of decay and they don't know the constant of proportionality. They do know the number of grams released. They also know that its half-life – the number of years required for half of the atoms in a sample to decay – is 24,100 years. You might think that knowing its half-life would tell you the rate of decay, but it doesn't – not by itself. You need more information.

Let's say they want to predict how many years it's going to take for the number of grams to decay down to 1 gram just by knowing that 10 grams were released. We'd have to have a way of determining the constant of proportionality. How would we do that?

For this case it's called the Exponential Growth and Decay model formula:

$$y(t) = Ce^{kt}$$

We know $y(t)$ is the number of grams released. That's a known quantity (10). Our ultimate goal is to solve for t where t is the time it takes to decay down to 1 gram. To do that we need to solve for k and we also need C.

So, do we know anything about a combination of a y and t that would go together that would help us eliminate some variables to isolate C or k? Let's look at the initial condition. That's always a favorite condition to look at. When time $t = 0$ what is y? 10 Right? So, we have the initial condition $t = 0$ and $y = 10$ that will give us a particular solution (heard that before?). Let's see what we can do with it:

What d'ya know we can solve for C:

$$\begin{aligned} 10 &= Ce^{k(0)} \\ &= Ce^{0} \\ 10 &= C \end{aligned}$$

So now we have our C for our particular solution. Now we've got to solve for k. Here is where we can use $t = 24{,}100$ (that's its half-life). How many grams would be left? Wouldn't $y = 5$ by then? Yes, it would. There would be 5 grams of plutonium left after 24,100 years. So, let's use that equation to solve for k:

$$\begin{aligned} 5 &= 10e^{k(24{,}100)} \\ \frac{1}{2} &= e^{24{,}100k} \\ \frac{1}{24{,}100}\ln\frac{1}{2} &= k \\ -0.000028761 &\approx k \end{aligned}$$

So now we can solve for t when 1 gram is left in our original equation because we know y and C and k. Let's do it:

$$\begin{aligned} 1 &= 10e^{-0.000028761t} \\ \frac{1}{10} &= e^{-0.000028761t} \\ \ln\frac{1}{10} &= -0.000028761t \\ \frac{\ln\frac{1}{10}}{-0.000028761} &= t \\ \frac{-2.302585}{-0.000028761} &= t \\ 80,059 &\approx t \end{aligned}$$

So, it's going to take 80,059 years for the 10 grams to decay to 1 gram and we found that out by using the Exponential Growth and Decay model formula.

In these problems, you typically will be solving for C and/or k first based on the given information before you get to the solution to the problem.

The Exponential Growth and Decay Model Summary

The book just gives you the formula. So now you should have an idea of how we got to the formula. It all has to do with that wild and crazy number e and the fact that it has that special relationship with the derivative of e^x equaling itself. Now you should know what it means when people say "My goodness, that thing is growing EXPONENTIALLY!"

Here's a summary statement of what the mathematicians discovered:

$y(t) = Ce^{kt}$ **is a solution to the differential equation** $\frac{dy}{dt} = ky$.

This is true for any value of the constant C.

Understanding this material is a matter of carefully going through each example in the book with paper and pencil so you can see how they arrive at each step in the process.

Videos

I have two videos for you today. Both show examples of solving first order separable differential equations which are the type of differential equations we are dealing with in Chapter 5. They are differential equations that can be solved by separation of variables. I'd like you to look at them before you try the homework problems. I'm showing you two videos because this type of problem is guaranteed to be on the AP Exam – guaranteed! It will be one of the Free Response Questions.

The first video with Patrick has an error in it at the end. You should recognize it. After he exponentiates both sides he forgets to cancel out the natural logarithm ln from the right side of the equation. There should be no ln in the final solution. There's a popup message pointing out the error.

Wrap Up

That's all I have for now. It's time for you to shuffle on over to WebAssign and get started.

Enjoy your day!

"No experiment is ever a complete failure. It can always be used as a bad example."

\- Paul Dickson