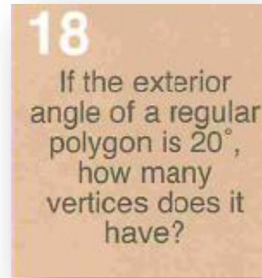
January

Leonhard Euler (1707-1783)



Hello!

Section 6.1 – Slope Fields and Euler's Method

Today we start Chapter 6. The first section is on Slope Fields and Euler's Method. So who was Leonhard Euler (pronounced *Oiler*)? Here's what Wikipedia has to say about him:

> "Euler made important discoveries in fields as diverse as calculus and graph theory. He also introduced much of the modern mathematical terminology and notation, particularly for mathematical analysis, such as the notion of a mathematical function.
>
> Euler is considered to be the preeminent mathematician of the 18th century and one of the greatest of all time."

Wow! Now that's what I call quite a biography – "the preeminent mathematician of the 18th century and one of the greatest of all time!" This guy must have been really, really smart! Whew!

So how does he figure in today's lesson? Well, Euler's Method is a numerical approach to approximating the particular solution of the differential equation, and today is all about differential equations.

So What Exactly is a Differential Equation?

Let's back up and review for a minute. What is a differential equation? It's nothing more than an equation containing a derivative. Actually, we have created many differential equations simply by finding derivatives of functions.

In this chapter, we'll start with the differential equation and work backwards to find the original equation. Now if you stop and think about it, all we'll be doing is applying integration methods. It should intuitively make sense that to solve a differential equation you would integrate it.

A Simple Example of a Differential Equation

Let's take the simplest example there is of a differential equation. By now you should be very familiar with position, velocity and acceleration. We know that velocity is the derivative of position (distance), and that acceleration is the derivative of velocity.

Position = $s(t)$ Velocity = $v(t) = s'(t)$ Acceleration = $a(t) = v'(t) = s''(t)$

Let's consider going "backward" from velocity to distance, for a car moving at 50 mph, at a constant velocity. How far does it go in a given time? You remember the formula for distance:

Distance = Rate x Time

Here's an example:

$$s\,(t) = 50\,t$$

where s is the distance of the car in miles, and t is the time in hours. Now let's go backwards and find the rate of motion, or rate of change (which is another way of saying *derivative*). Here's the equation for that:

$$\frac{ds}{dt} = s'(t) = 50\,mph$$

Solution to the Differential Equation

This is called a *differential equation* for the function s. If we were asked to "solve" this differential equation it would mean find the original function $s(t)$, or another way of putting it, find the antiderivative, which would look like this:

$$s(t) = \int 50\,dt = 50t + C$$

Particular Solution to the Differential Equation

Furthermore, if we know the value of s and t we can solve for the value of C. For instance, if $t = 2$ hours, and the position $s = 25$ miles from the initial position, then the constant C would equal -75.

This means the car must have started 75 miles before the fixed point, and gone 25 miles beyond it in 2 hours, in order to travel 100 miles.

$$25 = 50\,(2) + C \quad ==> \quad -75 = C$$

So [50 (2)], which is (rate x time), is the total distance, and 25, which is $s(t)$, is the current position. This is a good example of the difference between total distance and distance away from an initial position. They won't be the same thing if the constant C is not 0. That's a good thing to remember, but that's not what we're studying today.

So for those values of $t = 2$ and $s = 25$ the particular solution would be:

$$s(t) = 50t - 75$$

How We Got Where We're At

In the beginning of the book, when we were talking about derivatives, and solving for derivatives, we were given equations that looked like this:

$$s(t) = t^2 + 3t + 2$$

And we were asked to find the derivative, which would be:

$$s'(t) = 2t + 3$$

Then we learned about going backwards using integration, so we were given equations like this:

$$s'(t) = 2t + 3$$

And we were asked to find the antiderivative (indefinite integral), which would be:

$$\int 2t + 3\, dt = 2\int t\, dt + \int 3\, dt = 2\frac{t^2}{2} + 3t = t^2 + 3t + C$$

Modeling Real Life Brings the Function and It's Derivative Together in One Equation

In real life things are not quite so simple. It turns out, when you model real life phenomena, in other words, real life situations, you end up with equations that contain both an unknown function, such as $f(x)$, and its derivative $f'(x)$, such as:

$f'(x) = x - f(x)$ another way of writing this is $y' = x - y$

These are both saying the same thing and they are both differential equations, because they have derivatives in them. This is a logical topic to follow differentiation and integration, because they involve both techniques. It was necessary for you to learn each technique separately, and it is logical that to model some things you need both of them in the same equation.

Solving Differential Equations Results in a Family of Solutions

You remember that finding antiderivatives, or indefinite integrals, resulted in a family of solutions, so we had to always add a constant at the end of a solution. It follows that you need to do the same thing with differential equations, because you are normally solving for the original function, $f(x)$ or y in the second format, you're going backwards or in the integral direction, so you need the constant.

Because in real life you're typically looking for a particular function, and you know a particular x and y value of the function you're looking for, you can plug in the specific values into the general solution, then solve for C, then plug C into the solution, and viola´, you've got the particular function you were trying to find.

Some Differential Equations Can't Be Solved or Are Very Difficult to Solve

Just like integrals, there are a lot of differential equations that cannot be solved, or are very difficult to solve using just equations, or what's called analytically. So some very smart mathematicians thought of a way to approach this problem graphically.

Obviously, if you could draw the graph of the functions in the family of solutions, then you would know the shape of the graph of the particular solution function you were looking for. But you can't do that, because you can't solve the equation.

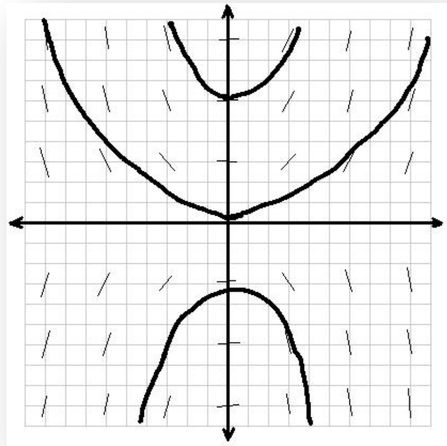
Sketching a Particular Solution Using a Slope Field

So this is where the graphs come into play. Here's what they figured out: if you found the slope at particular points and drew short segments at those points, and you did this in all four quadrants of the *xy* plane of a graph, then you could get a pretty good idea of the direction of the family of solutions. Then you could take the particular values you had for x and y, and find the particular curve that point fell on, and that curve would be your particular solution curve.

Now, I'm saying "curve" in a loose sort of way, because it's not really a curve, it's a bunch of small segments. But you can get a pretty good idea of the direction of the curve. If you put all these short segments from the family of solutions together you have what is called a "slope field."

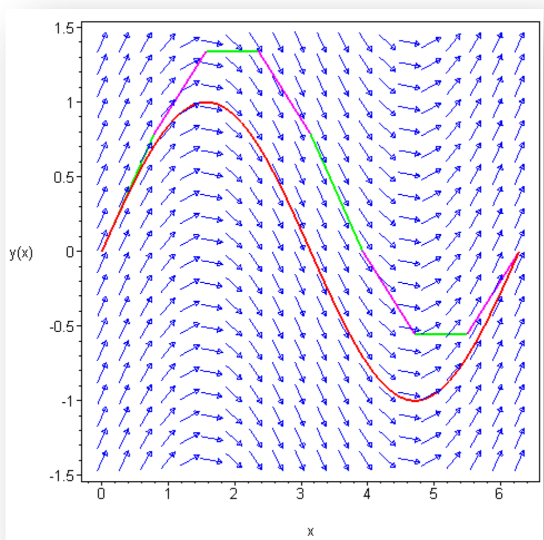
You can find the slope of these short segments because you know the derivative, cuz what's the derivative? It's the slope. I know, you already knew that. I'm just saying it because it's so neat that they figured out how to use this idea, those very smart mathematicians I just mentioned.

So what does a slope field look like? Here are some examples showing a slope field with particular solutions drawn in on top of them:



The short segments in the graph represent the slopes at points of the family of solutions to the given differential equation (whatever it is). The curves that are drawn in are the particular solutions amongst the family of solutions.

Here's another example:



Growth and Decay is an Application of Differential Equations

The book introduces what will turn out to be a very practical application of differential equations. It involves our old friend, that wild and crazy number, *e*. You'll see that number pop up in all kinds of places in all kinds of applications of math. That's what makes it so wild and crazy. The particular applications that we'll be studying are growth and decay. I'll be talking more about them in the next couple of days.

Euler's method

Euler's Method will not be on the AP Calculus AB Exam, but I had you read about it for three reasons: 1) he's a very important person in the history of mathematics, 2) you'll study it in the next course of college calculus, if you take it in college, and 3) it's not that hard of a subject. But if you don't understand it, you won't have to worry about it for the Exam.

Wrap Up

That's all I have for today. It's time for you to go to the Current Assignments page and get started on your assignment to see how well you understand this new and exciting (I hope) application of what you've studied so hard to learn, differentiation and integration.

Enjoy your day!!

We judge ourselves by what we feel capable of doing, while others judge us by what we have already done. Henry Wadsworth Longfellow